Continuous Time Fourier Transform

Fourier series can deal only with the periodic signals. This is the major drawback of Fourier series. It can't process naturally produced aperiodic signals.

Fourier Transform ~~limits~~ ~~the~~ overcome the limitation of the Fourier series. It is a transformation technique which transforms signals from continuous time domain to the corresponding frequency domain and vice versa and which applies for both periodic as well as aperiodic signals.

Uses:- Fourier transform is extensively used in the ~~analysis~~ analysis of linear time-invariant system (LTI), ~~cryp~~ cryptography, signal analysis, signal processing, astronomy etc.

Let us consider a continuous time periodic signal $x_T(t)$ having period T and $x(t)$ be the non-periodic signal ~~Then~~, when ~~∞~~

$T \to \infty$

Then,

$$x(t) = \lim_{T \to \infty} x_T(t) \quad \text{--- (I)}$$

Since, the Fourier series of a aperiodic signal $x(t)$ is given by Exponential Fourier series,

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt \quad \text{--- (II)}$$

$$\omega_0 = \frac{2\pi}{T}$$

C_n also represents the amplitude of component of frequency

$\therefore$ The equation (II) takes the forms

$$T C_n = \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt$$

Let $n\omega_0 = \omega$, at $T \to \infty$

Then $\omega_0 = \frac{2\pi}{T} \to 0$.

and the discrete Fourier spectrum becomes continuous.

$x_T(t) \to x(t)$

So,
$$T C_n = \lim_{T \to \infty} \int_{-T/2}^{T/2} x_T(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} \left[\lim_{T \to \infty} x_T(t) \right] \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt$$

$$T C_n = X(\omega) \longrightarrow \text{(III)}$$

So,
$$\boxed{X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} dt}$$

This is called the Fourier Transform of the signal $x(t)$. It also ~~repres~~ represents the frequency spectrum of $x(t)$ and called spectral density function.

Now, $\because$
$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t} \longrightarrow \text{(IV)}$$

$$\because C_n = \frac{T C_n}{T} = \frac{X(\omega)}{T} = \frac{X(\omega) \cdot \omega_0}{2\pi} \quad \left[\because T = \frac{2\pi}{\omega_0}\right.$$

On substituting the value of C_n in equation (IV) we get,

$$x_T(t) = \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t} \cdot \omega_0.$$

Now taking the limit $\lim_{T \to \infty}$ on the both sides we get.

$$\lim_{T\to\infty} x_T(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} \cdot e^{jn\omega_0 t} \cdot \omega_0$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega) \cdot e^{jn\omega_0 t} \, d\theta$$

As $T \to \infty$

then $\omega_0 = \frac{2\pi}{T} \to 0$, becomes infinitesimally small and may be represented by $d\omega$. also, the summation becomes integration.

$$\therefore \lim_{T\to\infty} x_T(t) = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \, d\omega.$$

So,

$x(t)$ is called the inverse Fourier transform of $X(\omega)$.

So, $$\boxed{F[x(t)] = X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt}$$

and $$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \cdot d\omega.}$$

Magnitude and phase Representation of Fourier Transform.

The magnitude and phase representation of the Fourier transform is the tool used to analyse the transformed signal.

In general $X(\omega)$ is a complex valued function of ω. Therefore,

$$X(\omega) = X_R(\omega) + jX_I(\omega).$$

$$X(\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

where
$X_R(\omega)$ = Real part
$X_I(\omega)$ = Imaginary part of $X(\omega)$.

The magnitude of $X(\omega)$ is given by

$$\boxed{|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}}$$

and phase of $X(\omega)$,

$$\boxed{\angle X(\omega) = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}}$$

The plot of $|X(j\omega)|$ or $|X(\omega)|$ versus ω is called amplitude spectrum and the plot of $\angle X(j\omega)$ versus ω is called phase spectrum. Both together is known as frequency spectrum.

Existence of Fourier Transform.

The Fourier transforms does not exists for all aperiodic functions. However those functions which satisfy the following condition are called Dirichlets condition are

1) The function $x(t)$ is absolutely integrable over the interval $-\infty$ to $+\infty$ i.e

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

2) $x(t)$ has a finite number of discontinuities in every finite time interval. and further each of these discontinuities must be finite

3) $x(t)$ has a finite number of maxima and minima in every finite time interval.

Fourier Transforms of some standard signals

i) Impulse function. [$\delta(t)$]

Soln: $\therefore$ $x(t) = \delta(t)$

$\therefore$ we know that,

$$\delta(t) = \begin{cases} 1 & \text{for } t=0 \\ 0 & \text{for } t \neq 0. \end{cases}$$

The Fourier Transform of $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t) \cdot e^{-j\omega t} \, dt$$

$\therefore$ at $t=0$. $\delta(t) = 1$

$$X(\omega) = \int_{-\infty}^{+\infty} 1 \, e^{-j\omega t} \, dt = \int_{-\infty}^{+\infty} 1 \cdot 1 dt \cdot [t]_{-\infty}^{+\infty} = 0 + \infty =$$

$$1[dt]_0^0 = 1[0+\infty]$$

$$X(\omega) = 1$$

Now magnitude of $X(\omega) = |X(\omega)| = \sqrt{X(\omega)^2 + X(j\omega)^2} = 1$

and phase $\angle X(j\omega) = \tan^{-1}\frac{0}{1} = 0 = 0.$

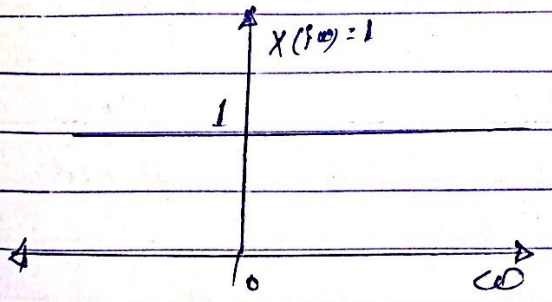


Fig: magnitude spectrum

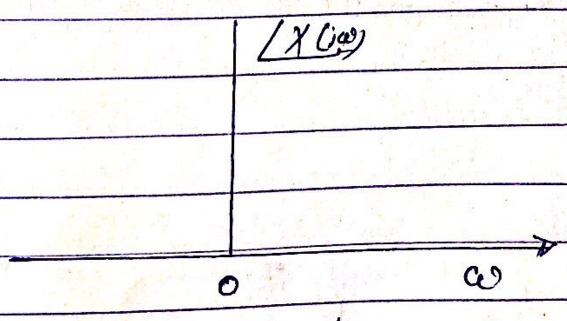


Fig: phase spectrum

II) Single-sided real exponential function $e^{-at}u(t)$.

Soln: $\because$ $x(t) = e^{-at}u(t)$.

$\because$ we know that,

$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-at}\cdot e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{-1}{(a+j\omega)}\cdot[0-1]$$

$$\therefore F[x(t)] = X(j\omega) = \frac{1}{(a+j\omega)}$$

Now $|X(j\omega)| = \sqrt{a^2+\omega^2}$

and phase $\angle X(j\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)$

$$X(\omega) = \frac{1}{(a+j\omega)}\times\frac{(a-j\omega)}{(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2}$$

$$= \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

~~also~~

The magnitude, $|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

also, $\angle X(j\omega) = -\tan^{-1}\frac{\omega}{a}$ for all ω.

plot of phase spectrum and magnitude spectrum

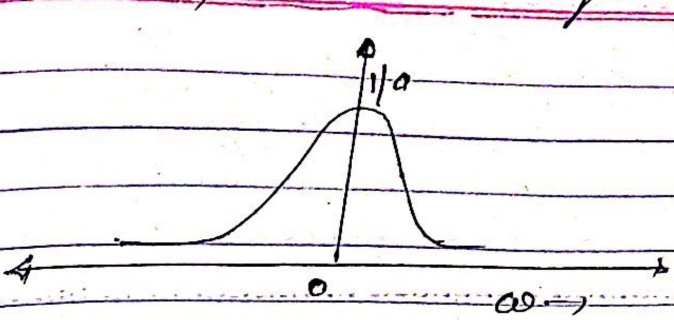
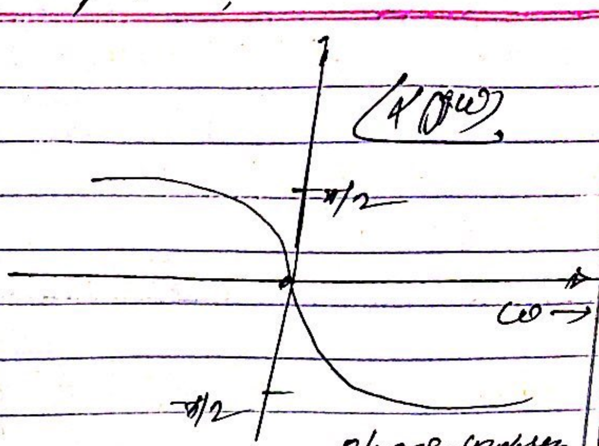


Fig: Amplitude spectrum



phase spectrum